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Dual statistical systems and geometrical string

George K. Savvidy [a,b], Konstantin G. Savvidy [c], Paul G. Savvidy [d]

[a] Physics Department, University of Crete, 71409 Iraklion, Crete, Greece
[b] Yerevan Physics Institute, 375036 Yerevan, Armenia
[c] Princeton University, Department of Physics, P.O. Box 708, Princeton, NJ 08544, USA
[d] Physics Department, University of Athens, 10680 Athens, Greece

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Abstract

We analyze a statistical system with an energy functional which is proportional to the linear size of the surfaces. We find a dual system, whose high temperature expansion generates random surfaces with a linear amplitude. In this model the independent variable is placed at the vertices of the 3D cubic lattice and takes four values. The interaction is nearest neighbour and anisotropic. These dual systems have the same relation to each other as the 3D Ising ferromagnet has to the 3D gauge spin system.

0. An alternative model of the string theory which is based on the extension of the Feynman integral over paths to an integral over surfaces, so that both amplitudes coincide in the cases when the surface degenerates into a single particle world line was suggested in Refs. [1,2]. To study the physical properties of this string with *linear action* it is important to find the field theoretical representation of the partition function in terms of local variables. In the recent articles [3,4] the authors formulated a new class of statistical systems, whose interface energy is associated with the edges of the interface. These systems have a geometrical representation of the low temperature partition function in terms of the sum over paths and surfaces, with an amplitude which is proportional to the total *curvature* of the path on the two-dimensional lattice and to the linear *size* of the surface in three dimensions [5,1,2,6–8].

This lattice implementation of the linear string crucially depends on the way we ascribe weights to self-intersections. It was suggested in Ref. [1] to assign special weights to self-intersections depending on a parameter k, which can be named the self-intersection coupling constant [4]. One can distinguish different cases through the value of k: (i) $k = 0$, the "bosonic" case, (ii) $0 < k < \infty$, "soft-fermionic" and (iii) $k = \infty$, the "fermionic" case [4]. The form of the Hamiltonian H^k and the symmetry of the system essentially depends on k: when $k \neq 0$ one can flip the spins on arbitrary parallel layers and thus the degeneracy of the ground state is equal to 3×2^N, where N is the size of the lattice. When $k = 0$ the system has even higher symmetry and the degeneracy of the ground state is equal to 2^{3N}. In terms of the Ising spin variables σ_r the Hamiltonian of the system $k = 0$ has the form [4]

$$H^{3D}_{\text{gonihedric}} = - \sum_{r,\alpha,\beta} \sigma_r \sigma_{r+\alpha} \sigma_{r+\alpha+\beta} \sigma_{r+\beta}, \tag{1}$$

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where $\boldsymbol{r}$ is a three-dimensional vector on the lattice Z^3, whose components are integer, and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are unit vectors parallel to the axes. We should stress that the Ising spins in (1) are on the vertices of the lattice Z^3 and are not on the links. In addition to the usual symmetry group Z_2 system (1) has an extra symmetry: one can independently flip the spins on *any* combination of planes, even on the intersecting ones. This system is a sort of supersymmetric point in the "space" of gonihedric Hamiltonians H^k. Our aim is to construct a Hamiltonian which is dual to $H^{3D}_{\text{gonihedric}}$ (1) and equally well describes surfaces with a linear amplitude. The enhanced symmetry of system (1) plays an essential role in our construction of the dual system. But let us first review the known results in the case of the Ising model.

1. The low temperature expansion of the Ising ferromagnet has a beautiful geometrical representation of the partition function in terms of the sum over the paths on the two-dimensional lattice and over the surfaces on the three-dimensional lattice. The corresponding amplitudes are proportional to the *length* of the paths and to the *area* of the surfaces. This geometrical representation allows one to show that the high temperature expansion of the 2D Ising ferromagnet is the same as the low temperature expansion and that the system is therefore self-dual [9–14]. In the case of the 3D Ising ferromagnet

$$H^{3D}_{\text{Ising}} = - \sum_{\text{links}} \sigma\sigma , \tag{2a}$$

the high temperature expansion does not coincide with its low temperature expansion and can be represented as a sum over the random walks with an amplitude which is proportional to the *length* of the paths.

In 1971 Wegner succeeded in constructing a three-dimensional spin system with local interaction, whose low temperature expansion coincides with the random walks with the *length* amplitude and whose high temperature expansion coincides with the sum over the random surfaces with the *area* amplitude. In this natural way he has found the gauge spin Hamiltonian which is dual to the 3D Ising ferromagnet [14]

$$H^{3D}_{\text{gauge}} = - \sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma . \tag{2b}$$

One motivation for the study of spin systems (1) is that they can help one to understand the dynamics of random surfaces with *area action* [15–21].

It is an important fact that there exists a dual representation of the random surfaces on the lattice with the area law in terms of Ising and Wegner Hamiltonians (2a) and (2b) [22–26]. Our aim is to construct a Hamiltonian which is dual to $H^{3D}_{\text{gonihedric}}$ (1) and to compare these complementary systems with each other. As we explained, the deep reason why this construction is possible lies in the enhanced symmetry of system (1). In the next sections we shall geometrically construct the system which is dual to the one defined by Hamiltonian (1).

2. The partition function of system (1) can be written in the form

$$Z(\beta) = \sum_{\{\sigma\}} \exp\left(-\beta H^{3D}_{\text{gonihedric}}\right). \tag{3}$$

As it was shown in Refs. [3,4], the low temperature expansion is extended over all closed surfaces $\{M\}$ with the restriction that only an *even* number of plaquettes can intersect at any given *edge* ($2r = 0, 2, 4$) and that there is only one plaquette at every given place

$$Z(\beta) = \sum_{\{M\}} \exp[-2\beta A(M)], \tag{4a}$$

where $A(M)$ is the linear-gonihedric action [1,2,5] which is equal to

$$A(M) = \sum_{\langle i,j \rangle} \lambda_{i,j} | \pi - \alpha_{i,j} |, \tag{4b}$$

where the summation is over all edges $\langle i, j \rangle$ of M, and $\alpha_{i,j}$ is the angle between two neighbour plaquettes of M in Z^3 having a common edge $\langle i, j \rangle$ of the length $\lambda_{i,j}$. It is easy to see that $A(M)$ is proportional to the linear size of the surface M [1,2,5] and there are no weights associated with intersections because we consider the case $k = 0$.

Hamiltonian (1) and partition function (3) provide the representation (4) of the randomly fluctuating surfaces on the lattice with gonihedric action $A(M)$ in terms of locally interacting fields-spins [3,4]. To construct the dual Hamiltonian we should find the geometrical representation of the high temperature expansion of the system (1), (3). For that let us consider the high temperature representation of (3),

$$Z(\beta) = \sum_{\{\sigma\}} \prod_{\text{plaquettes}} \operatorname{ch} \beta[1 + \operatorname{th} \beta(\sigma\sigma\sigma\sigma)]. \quad (5)$$

Expanding the product and summing over $\{\sigma\}$ one can see that only the terms which contain an even number of plaquettes at every given *vertex* produce a nonzero contribution, therefore

$$Z(\beta) = (2 \operatorname{ch} \beta)^{3N^3} \sum_{\{\Sigma\}} (\operatorname{th} \beta)^{s(\Sigma)}, \quad (6)$$

where the summation is extended over all surfaces $\{\Sigma\}$ with an even number of plaquettes at any given vertex. $s(\Sigma)$ is the number of plaquettes of Σ, e.g. the area of the surface. Open surfaces are allowed. In the next section we will describe in detail this set of surfaces $\{\Sigma\}$, and the concept of group structure on this set will be introduced.

3. As we have seen, the high temperature expansion (6) of the gonihedric system (1), (3) in 3D is extended over surfaces $\{\Sigma\}$ which can be considered as a collection of plaquettes on a cubic lattice with the restriction that only an even number of plaquettes can intersect at any given *vertex* and that there is no more than one plaquette at every given place.

Let us attach plaquette variables U_P to each plaquette P of Z^3

$$U_P = -1 \quad \text{if} \quad P \in \Sigma \quad \text{and} \quad U_P = 1 \quad \text{if} \quad P \notin \Sigma. \quad (7)$$

There are twelve plaquettes P incident to every vertex of the lattice. The constraint on the plaquette variables U_P in every vertex

$$\prod_{\text{12 plaquettes incident to vertex}} U_P = 1, \quad (8)$$

uniquely characterizes our set of surfaces $\{\Sigma\}$ in (6).

Now one can introduce the group structure on this set of surfaces $\{\Sigma\}$ (8). Let us consider two surfaces Σ^1 and Σ^2 and denote their plaquette variables as U_P^1 and U_P^2 respectively. Let us define the group product of these two surfaces as

$$U_P = U_P^1 U_P^2. \quad (9)$$

According to this definition a given plaquette belongs to a group product of two surfaces $\Sigma = \Sigma^1 \otimes \Sigma^2$ only if it belongs to exactly one of them: to Σ^1 or to Σ^2.

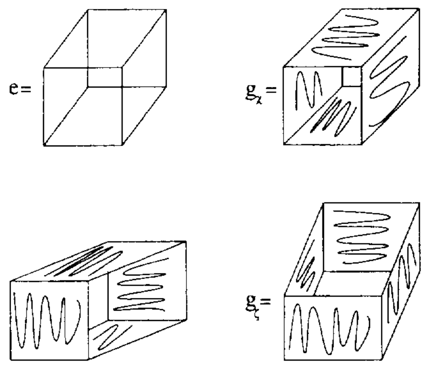

Fig. 1. The elementary "matchbox" surfaces (10a).

One should check that the group product defined in this way leaves the surfaces in the same class (8). Indeed if (8) holds for Σ^1 and Σ^2 then it holds for the surface product Σ, that is Σ also has an even number of plaquettes on every vertex (8). The inverse element of Σ coincides with itself. The set of surfaces $\{\Sigma\}$ (8) finally forms an Abelian group G.

One can show that the whole group G is a product of the local group G_ξ. This group G_ξ has four elements – elementary "matchbox" surfaces,

$$e, \quad g_\chi, \quad g_\eta, \quad g_\varsigma \quad (10a)$$

in Fig. 1 and with the multiplication table

$$e g_{\chi,\eta,\varsigma} = g_{\chi,\eta,\varsigma}, \quad g_\chi g_\chi = g_\eta g_\eta = g_\varsigma g_\varsigma = e,$$
$$g_\chi g_\eta = g_\varsigma, \quad \text{etc.}, \quad (11)$$

which follows from the multiplication law (9) and the definition of the matchbox surfaces; see Fig. 1.

With the help of G_ξ one can reconstruct any surface Σ of the set $\{\Sigma\}$ (8). Indeed any set of elementary matchbox surfaces e, g_χ, g_η, g_ς (10), (11) distributed independently over the lattice Z^3 describes some allowed surface Σ and any given surface from $\{\Sigma\}$ (8) can be decomposed into the product of G_ξ

$$\Sigma = \prod_\xi G_\xi. \quad (12)$$

This approach allows us to describe the original surface Σ in terms of a new independent matchbox spin variable

$$G_\xi = \{e(\xi), g_\chi(\xi), g_\eta(\xi), g_\varsigma(\xi)\}, \quad (10b)$$

which should be attached to the centers of the cubes ξ of the original lattice Z^3, e.g. to the vertices of the dual lattice Z^{*3}.

This is what we aimed at: to express the surface configuration Σ with the constraints (8) in terms of the independent local variable G_ξ. The group G_ξ is an Abelian group of the fourth order and therefore has four one-dimensional irreducible representations

$$E = \{1, 1, 1, 1\}, \quad R^{\chi} = \{1, 1, -1, -1\},$$
$$R^{\eta} = \{1, -1, 1, -1\}, \quad R^{\varsigma} = \{1, -1, -1, 1\}, \tag{13a}$$

with the orthogonality relations

$$\sum_{G_\xi} R^{l}(G_\xi) R^{m}(G_\xi) = 4\delta^{l,m} \quad (l, m = \chi, \eta, \varsigma),$$
$$\sum_{G_\xi} R^{\chi}(G_\xi) R^{\eta}(G_\xi) R^{\varsigma}(G_\xi) = 4. \tag{13b}$$

We will use these representations to express algebraically the matchbox spin variable G_ξ. The next step is to express the amplitude of Σ in terms of these independent variables and to construct the dual Hamiltonian.

4. In the high temperature expansion (6) the energy of the surface Σ is equal to the number of plaquettes $s(\Sigma)$, that is to the area. We would now like to construct a new system of locally interacting matchbox spins G_ξ with identical low temperature expansion. For that we should properly organize the local interaction of the matchbox spins G_ξ.

The dual Hamiltonian is nonhomogeneous in the directions χ, η, and ς and is equal to

$$H_{\mathrm{dual}} = \sum_{\xi} \left(H_{\xi,\xi+\chi} + H_{\xi,\xi+\eta} + H_{\xi,\xi+\varsigma} \right), \tag{14}$$

where χ, η, and ς are unit vectors in the corresponding directions of the dual lattice and

$$H_{\xi,\xi+\chi} \equiv H(G_\xi, G_{\xi+\chi}) = -R^{\chi}(\xi) R^{\chi}(\xi+\chi),$$
$$H_{\xi,\xi+\eta} \equiv H(G_\xi, G_{\xi+\eta}) = -R^{\eta}(\xi) R^{\eta}(\xi+\eta),$$
$$H_{\xi,\xi+\varsigma} \equiv H(G_\xi, G_{\xi+\varsigma}) = -R^{\varsigma}(\xi) R^{\varsigma}(\xi+\varsigma). \tag{15}$$

Here we abreviated G_ξ by simply ξ, that is $R(G_\xi) \equiv R(\xi)$, a notation that should make no difference. The partition function of the dual system (14), (15) can be written in the form

$$Z(\beta^*) = \sum_{\{G_\xi\}} \exp(-\beta^* H_{\mathrm{dual}}). \tag{16}$$

Now we should check that the low temperature expansion of the dual system (16) indeed coincides with the high temperature expansion of the original system (6). We will see that this indeed takes place, and we can expect that high temperature expansion of the dual Hamiltonian will provide us with the new lattice representation of the random surfaces with gonihedric action. In the next section we will show that the two systems (1) and (14), (15) are indeed dual to each other.

5. Let us define the surface of the interface Σ for the matchbox spins G_ξ in the following way: plaquette P belongs to Σ if the product of the neighboring matchbox spins is equal to -1,

$$P_{\xi,\xi+\chi} \in \Sigma \quad \text{if} \quad R^{\chi}(G_\xi) R^{\chi}(G_{\xi+\chi}) = -1 \tag{17}$$

and

$$P_{\xi,\xi+\chi} \notin \Sigma \quad \text{if} \quad R^{\chi}(G_\xi) R^{\chi}(G_{\xi+\chi}) = 1. \tag{18}$$

In the same way we should define $P_{\xi,\xi+\eta}$ and $P_{\xi,\xi+\varsigma}$. These surfaces are of the class $\{\Sigma\}$ (8) because the plaquette variables U_P defined as

$$U_{\xi,\xi+\chi} = R^{\chi}(G_\xi) R^{\chi}(G_{\xi+\chi}),$$
$$U_{\xi,\xi+\eta} = R^{\eta}(G_\xi) R^{\eta}(G_{\xi+\eta}),$$
$$U_{\xi,\xi+\varsigma} = R^{\varsigma}(G_\xi) R^{\varsigma}(G_{\xi+\varsigma}), \tag{19}$$

identically resolve the constraints (8). The correspondence between matchbox spin configurations and surfaces Σ is four-to-one, instead of two-to-one in the case of surfaces of interface of the Ising ferromagnet.

By construction the energy of the surface Σ of matchbox spin interface is equal to its area and therefore the low temperature expansion of the dual system (16) indeed coincides with the high temperature expansion of the original system (6).

6. As the last step in this construction we should prove that the high temperature expansion of the dual system (14), (15) is equivalent to the sum over random surfaces with an amplitude which is proportional to the linear size of the surface $A(M)$ and coincides with the low temperature expansion (4) of (3).

For that let us consider the high temperature representation of the partition function (16)

$$Z(\beta^*) = \sum_{\{R(\xi)\}} \prod_{\xi} (\mathrm{ch}\ \beta^*)^3 \left(1 - \mathrm{th}\ \beta^* H_{\xi,\xi+\chi}\right) \times \left(1 - \mathrm{th}\ \beta^* H_{\xi,\xi+\eta}\right)\left(1 - \mathrm{th}\ \beta^* H_{\xi,\xi+\varsigma}\right). \tag{20}$$

Expanding the product we will have terms of the form

$$\sum_{\{R(\xi)\}} \prod_{\text{over paths on } Z^{*3}} R(\xi), \tag{21}$$

representing the chains of R's (13) of different lengths on the lattice Z^{*3}. Let us see which of these terms are nonzero. Using the relations

$$\begin{aligned} R^{\chi}(\xi)R^{\chi}(\xi) &= R^{\eta}(\xi)R^{\eta}(\xi) \\ &= R^{\varsigma}(\xi)R^{\varsigma}(\xi) = E, \\ R^{\chi}(\xi)R^{\eta}(\xi)R^{\varsigma}(\xi) &= E, \end{aligned} \tag{22}$$

which hold for the irreducible representations (13), one can obtain

$$\begin{aligned} &\sum_{G_\xi} H_{\xi-\chi,\xi} H_{\xi,\xi+\chi} = 4R^{\chi}(\xi-\chi)R^{\chi}(\xi+\chi), \\ &\sum_{G_\xi} H_{\xi-\chi,\xi} H_{\xi,\xi+\eta} = \sum_{G_\xi} H_{\xi-\chi,\xi+\varsigma} = 0. \end{aligned} \tag{23}$$

The last two relations tell us that any loop which contains one turn at a right angle is equal to zero. We have three extra nonzero elementary vertices

$$\begin{aligned} &\sum_{G_\xi} H_{\xi,\xi+\chi} H_{\xi,\xi+\eta} H_{\xi,\xi+\varsigma} = -4R^{\chi}(\xi+\chi)R^{\eta}(\xi+\eta) \\ &\qquad \times R^{\varsigma}(\xi+\varsigma), \\ &\sum_{G_\xi} H_{\xi-\chi,\xi} H_{\xi,\xi+\chi} H_{\xi-\eta,\xi} H_{\xi,\xi+\eta} \\ &\quad = 4R^{\chi}(\xi-\chi)R^{\chi}(\xi+\chi)R^{\eta}(\xi-\eta) \\ &\qquad \times R^{\eta}(\xi+\eta), \\ &\sum_{G_\xi} H_{\xi-\chi,\xi} H_{\xi,\xi+\chi} H_{\xi-\eta,\xi} H_{\xi,\xi+\eta} H_{\xi-\varsigma,\xi} H_{\xi,\xi+\varsigma} \\ &\quad = 4R^{\chi}(\xi-\chi)R^{\chi}(\xi+\chi)R^{\eta}(\xi-\eta) \\ &\qquad \times R^{\eta}(\xi+\eta)R^{\varsigma}(\xi-\varsigma)R^{\varsigma}(\xi+\varsigma) \end{aligned} \tag{24}$$

and all other vertices are equal to zero.

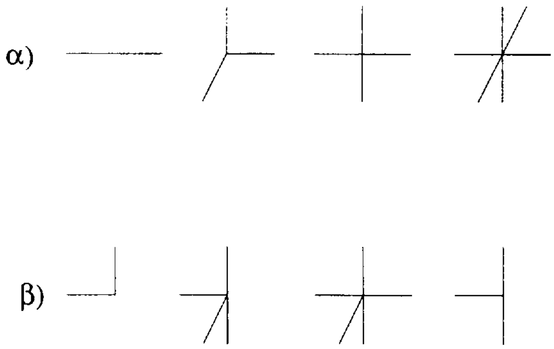

Fig. 2. (α) The four nonzero vertices of the type (23) and (24), which are permitted in the construction of the "skeleton". (β) The forbidden vertices.

Thus the high temperature expansion of the dual system (14), (15) contains only those loops on the lattice Z^{*3} which have only one of those four nonzero vertices of the type (23) and (24). One can imagine every term of this expansion as a "skeleton" constructed by the loops of "bones" restricted by the constraints (23) and (24). The amplitude is proportional to the total length of these bones. The partition function therefore has the form

$$Z(\beta^*) = \sum_{\text{skeletons}} (\mathrm{th}\ \beta^*)^{A(\text{skeleton})} \tag{25}$$

where A(skeleton) is the total length of the bones.

A nontrivial fact consists in the statement that one can "dress up" these skeletons by plaquettes so that

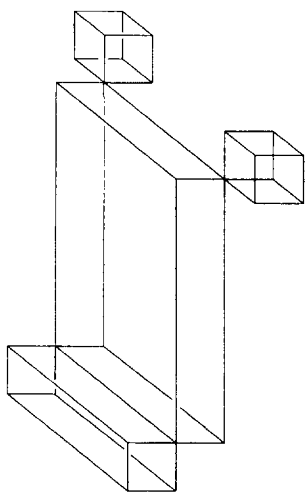

Fig. 3. Example of a "skeleton" in the high temperature expansion of the dual system (16).

bones will appear as the right angle edges of the surface M. In that case the A(skeleton) becomes identically equal to $A(M)$ and the summation over the skeletons reduces to the summation over surfaces M with linear-gonihedric action $A(M)$. This identification is possible only because the bones can join together and form the loops only though the vertices (23) and (24). So we have

$$\ln(\text{th}\ \beta^*) = -2\beta \tag{26}$$

and both systems (1) and (13), (14) and (15) are dual to each other in the same way as the 3D Ising ferromagnet (2a) is dual to the 3D Wegner gauge Hamiltonian (2b). They are complementary to each other in the sense that (2b) describes random surfaces with the area law and (14), (15) with the linear-gonihedric law.

Here the question is raised whether the system exhibits a phase transition. From the fact that the energy functional is proportional to the linear size of the surfaces one can expect that the system will show a phase transition in the 3D case which should be of the same nature as in the case of the 2D Ising ferromagnet.

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